Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Geometry and Physics 56 (2006) 1144–1174

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

On spacelike hypersurfaces of constant sectional curvature lorentz manifolds

A. Caminha*

Departamento de Matemática, Universidade Federal do Ceará, CEP 60455-760, Campus do Pici, Bl.914, Fortaleza, CE, Brazil

Received 8 April 2005; accepted 22 June 2005
Available online 2 August 2005

Abstract

Let $x : M^n \to \bar{M}^{n+1}$ be an n-dimensional spacelike hypersurface of a constant sectional curvature Lorentz manifold $\bar{M}$. Based on previous work of S. Montiel, L. Alías, A. Brasil and G. Colares studied what can be said about the geometry of M when $\bar{M}$ is a conformally stationary spacetime, with timelike conformal vector field K. For example, if M^n has constant higher order mean curvatures H_r and H_{r+1}, they concluded that M^n is totally umbilical, provided $H_{r+1} \neq 0$ on it. If $\mathrm{div}(K)$ does not vanish on M^n they also proved that M^n is totally umbilical, provided it has, a priori, just one constant higher order mean curvature.

In this paper, we compute $L_r(S_r)$ for such an immersion, and use the resulting formula to study both r-maximal spacelike hypersurfaces of $\bar{M}$, as well as, in the presence of a constant higher order mean curvature, constraints on the sectional curvature of M that also suffice to guarantee the umbilicity of M. Here, by L_r we mean the linearization of the second order differential operator associated to the r-th elementary symmetric function S_r on the eigenvalues of the second fundamental form of x.

MSC: 53C42 (primary); 53B30; 53C50; 53Z05; 83C99 (secondary)

Keywords: Lorentz manifolds; Spacelike hypersurfaces; Higher order mean curvatures; Rigidity problems

* Tel.: +55 85 4008 9885; fax: +55 85 4008 9889.
E-mail addresses: antonio_caminha@yahoo.com.br, caminha@mat.ufc.br.

0393-0440/$ – see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2005.06.007

1. Introduction

In the past 30 years, there has been an increasing interest in studying the structure of spacelike hypersurfaces of Lorentz manifolds of constant sectional curvature. This goes back to 1976, when S.Y. Cheng and S.T. Yau proved ([8]) the Calabi–Bernstein conjecture concerning complete maximal spacelike hypersurfaces of the Lorentz–Minkowsky space, namely, that the only ones are the spacelike hyperplanes.

For the De Sitter space, A.J. Goddard conjectured in [11] that complete spacelike hypersurfaces having constant mean curvature should be totally umbilical. Although the original problem turned out to be false in general, the efforts to prove it motivated a great deal of work by several authors, trying to figure out what additional geometric restrictions should be imposed in the hypersurface to get an affirmative answer. Goddard's conjecture was eventually proved to be true for the case of closed hypersurfaces, due to independent work of S. Montiel ([16]) and J.L. M. Barbosa and V. Oliker ([6]).

In recent years, the main stream of investigation has turned towards more general classes of Lorentz ambient spaces, dealing mostly with the problems of existence and uniqueness of constant mean curvature spacelike hypersurfaces. In [4], the authors proved that the only closed spacelike hypersurfaces of generalized Robertson–Walker spacetimes satisfying a suitable condition are the totally umbilical ones. By such spaces we mean warped products $I \times_f F^n$, where $I \subset \mathbb{R}$ is an open interval with the metric $-\mathrm{d}t^2$, F^n is an n-dimensional Riemannian manifold and $f : I \to \mathbb{R}$ is a positive smooth function. Note the these include both the Lorentz–Minkowsky space and the De Sitter space. Later on, S. Montiel considered (in [17]) the same problem for conformally stationary spacetimes, that is, Lorentz manifolds possessing a closed conformal timelike vector field K, where by closed we mean that the dual one form ω^K of K is closed. This class of spaces includes the previous one, for $K = f\frac{\partial}{\partial t}$ is a closed conformal timelike vector field in $I \times_f F^n$.

Lately, in [3], the authors studied what can be said about the geometry of a closed spacelike hypersurface M^n of a conformally stationary spacetime $\overline{M}^{n+1}$ if one imposes constraints on higher order mean curvatures of M. Among other results, they proved that if M is contained in a region of $\overline{M}$ where the divergence of the timelike conformal vector field K does not vanish, then M is totally umbilical provided it has, a priori, just one constant higher order mean curvature. In the De Sitter space, for example, this amounts for M being contained in the future or chronological past of an equator, thus agreeing with previous results in the literature. They also proved that M is totally umbilical provided it has two consecutive constant higher order mean curvatures H_r and H_{r+1}, with $H_{r+1} \neq 0$ on it (actually, this hypothesis is missing there).

Their method, which consists in applying certain integral formulae involving the higher order mean curvatures of M together with the classical Newton's inequalities (see [13]), has the disadvantage of not working for complete hypersurfaces. Moreover, in either the complete or compact case, asking what could be said of M once one has dropped the condition of the nonvanishing of the divergence of K is a question that naturally arises at this point. In particular, what can be said of r-maximal spacelike hypersurfaces of $\overline{M}$?

In this paper we give partial answers to these questions. Our approach is to compute $L_r(S_r)$ for a spacelike hypersurface $x : M^n \to \overline{M}^{n+1}$ of a time-oriented Lorentz manifold with no additional tructure, applying the resulting formulae in the study of the case of one constant higher order mean curvature. Here, by L_r we mean the linearization of the second order differential operator associated to the r-th elementary symmetric function S_r on the eigenvalues of the second fundamental form A of x. We also rely on a version of the Newton's inequalities slightly more general than that in [13].

The above machinery is put to work in order to show that a closed spacelike hypersurface of a time-oriented Lorentz manifold of constant sectional curvature $c \geq 0$, having constant scalar curvature R satisfying $c\left(\frac{n-2}{n}\right) < R \leq c$, is totally umbilical. This is also shown to be the case for complete spacelike hypersurfaces having constant scalar curvature R satisfying $c\left(\frac{n-2}{n}\right) < R < c$, once their mean curvature is nonnegative and attains a global maximum.

For general $r \geq 2$, a closed spacelike hypersurface M of a time-oriented Lorentz manifold of constant sectional curvature $c > 0$, having one constant higher order mean curvature $H_r \neq 0$, is also totally umbilical provided its sectional curvature K_M satisfies $0 < K_M \leq c$. This alternative condition works as a substitute for the nonvanishing of the divergence of the timelike vector field K, as discussed above. Moreover, for generalized Robertson–Walker spacetimes $I \times_f F^n$ of constant sectional curvature $c > 0$, it implies (according to [17]) that a closed hypersurface M satisfying the above hypotheses is necessarily of the form $\{t\} \times F$, for some $t \in I$; even more particularly, those are round spheres in the De Sitter space.

A sort of weak extension of the Cheng–Yau theorem mentioned in the beginning is also given. More precisely, if $x : M^n \to \overline{M}_c^{n+1}$ denotes a spacelike hypersurface of a time-oriented Lorentz manifold of constant sectional curvature $c \geq 0$, for which $H_r = 0$ and H_{r+1} is constant, then $H_j = 0$ on M for all $r \leq j \leq n$. This, in turn, gives the lower bound $n - r + 1$ for the index of relative nullity ([10]) of x, so that if $\overline{M}$ is the Lorentz–Minkowski space $\mathbb{L}^{n+1}$ and M is complete, then through every point of M there passes an $(n - r + 1)-$hyperplane of $\mathbb{L}^{n+1}$, totally contained in M.

A stronger result is true in the compact case, namely, that the condition $H_r = 0$ on M suffices to imply $H_j = 0$ on M for all $r \leq j \leq n$. Finally, for the case $c \leq 0$, a kind of Simon's integral formula (see [21]) is available: if $H_r = 0$ on M, then

$$\int_M \mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})]\mathrm{d}M \leq 0.$$

Moreover, if $H_{r+1} \neq 0$ then $\mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})] \geq 0$ on M gives $\nabla A = 0$ and H_{r+1}, H_{r-1} constant on M.

This paper is organized in the following manner: in Section 2 we establish some notation and recall several results needed for further developments. Then, in Section 3, we obtain the formula for $L_r(S_r)$ as a corollary of the more general computation of $L_q(S_r)$. Finally, in Section 4, we state and prove the applications referred to in the above paragraphs.

2. Preliminaries

Unless stated otherwise, M^n denotes a Riemannian manifold with Riemannian metric $g = \langle , \rangle$, Levi–Civitta connection ∇ and curvature tensor R;

$\mathcal{D}(M)$ denotes the commutative ring of smooth (i.e., C^∞) real functions on M.

2.1. Tensor fields

Let $\phi = \langle T\cdot, \cdot\rangle$ denote an arbitrary 2-tensor on M, and $\nabla\phi$ and $\nabla^2\phi = \nabla(\nabla\phi)$ denote its first and second covariant differentials. For each $V \in \mathcal{X}(M)$, it is easily verified that the recipe

$$(\nabla_V\phi)(X, Y) = (\nabla\phi)(X, Y, V)$$

defines another 2-tensor on M, the covariant derivative of ϕ in the direction of V. If $\nabla_V T$ denotes the linear operator associated to $\nabla_V\phi$, it is also easy to verify that

$$(\nabla_V T)(X) = \nabla_V(TX) - T(\nabla_V X).$$

Let $\{e_i\}$ be a moving frame on an open neighborhood $U \subset M$, with coframe $\{\omega_i\}$ and connection 1-forms ω_{ij}. Letting ϕ_{ij}, ϕ_{ijk} and ϕ_{ijkl} denote the components of ϕ, $\nabla\phi$ and $\nabla^2\phi$ with respect to $\{e_i\}$, the following relations take place:

$$\sum_k \phi_{ijk}\omega_k = \mathrm{d}\phi_{ij} - \sum_k \phi_{kj}\omega_{ik} - \sum_k \phi_{ik}\omega_{jk}; \tag{1}$$

$$\sum_l \phi_{ijkl}\omega_l = \mathrm{d}\phi_{ijk} - \sum_l \phi_{ljk}\omega_{il} - \sum_l \phi_{ilk}\omega_{jl} - \sum_l \phi_{ijl}\omega_{kl}. \tag{2}$$

The proof of the following lemma can be found in [9].

Lemma 1. *Let ϕ be a 2-tensor on M. With respect to an arbitrary moving frame $\{e_k\}$ on M, and letting $R_{irkl} = R(e_i, e_r, e_k, e_l)$, one has*

$$\phi_{ijkl} - \phi_{ijlk} = -\sum_r \phi_{rj}R_{irkl} - \sum_r \phi_{ir}R_{rjlk}.$$

The following remarks on components of tensors with respect to a given moving frame will be used in the next section.

Remark 1. A moving frame $\{e_k\}$ on (an open neighborhood of) M is called geodesic at p when $(\nabla_{e_k} e_i)(p) = 0$ for all $1 \le i, k \le n$, which is in turn equivalent to $\omega_{ij}(p) = 0$ for all $1 \le i, j \le n$. The usual way to build frames on M geodesic at $p \in M$ is by fixing a normal neighborhood of p and parallel transporting the elements of an arbitrary orthonormal basis of T_pM along the geodesic rays issuing from p. Whenever we speak of a frame on M, geodesic at some point $p \in M$, we will always assume that it has been built this way.

Remark 2. Note also that, for fixed $1 \le k \le n$, the above recipe gives $(\nabla_{e_k} e_i)(q) = 0$, for every $1 \le i \le n$ and every point q along the geodesic ray issuing from p with velocity vector e_k. Therefore, $\omega_{ij}(q)(e_k) = 0$ for all such i, j and q, and setting $\phi_{ij;k} = e_k(\phi_{ij})$ and

$\phi_{ij;kk} = e_k(e_k(\phi_{ij}))$ one has, along the geodesic ray issuing from p with velocity vector e_k, that

$$\phi_{ijk} = \phi_{ij;k} \quad \text{and} \quad \phi_{ijkk} = \phi_{ij;kk}. \tag{3}$$

The first part of (3) follows from (1), while the second one from substituting the first one into (2).

Remark 3. A 2-tensor ϕ on M is Codazzi when $\phi_{ijk} = \phi_{ikj}$ for all $1 \leq i, j, k \leq n$, and with respect to any moving frame $\{e_k\}$ on M. If this is the case, changing indices j and k in (2) gives

$$\phi_{ijkl} = \phi_{ikjl}, \quad \text{for all} 1 \leq i, j, k, l \leq n. \tag{4}$$

A 2-tensor ϕ on M is symmetric if $\phi(X, Y) = \phi(Y, X)$ for all $X, Y \in \mathcal{X}(M)$, or equivalently, when its associated linear operator T is self-adjoint. If $X \in \mathcal{X}(M)$, then $\nabla_X \phi$ is symmetric whenever ϕ is symmetric, so that $\nabla_X T$ is self-adjoint whenever T is self-adjoint. With respect to an arbitrary moving frame $\{e_k\}$ on M, the symmetry of ϕ is equivalent to $\phi_{ij} = \phi_{ji}$, for all $1 \leq i, j \leq n$. We define the squared norm of a symmetric 2-tensor ϕ on M by setting

$$|\phi|^2 = \mathrm{tr}(T^2) = \sum_{i,j} \phi_{ij}^2,$$

where tr denotes the *trace* of its associated linear operator T.

2.2. *Lorentz manifolds and isometric immersions*

Let $(\bar{M}^{n+1}, g)$ denote an $(n+1)-$dimensional, time-oriented Lorentz manifold, i.e., a Lorentz manifold with a timelike vector field K globally defined on it. A particular class of such manifolds is given by the conformally stationary Lorentz manifolds i.e., those for which the vector field K above can be chosen to be conformal, in the sense that

$$\mathcal{L}_K g = 2\phi g$$

for some $\phi \in \mathcal{D}(\bar{M})$, where $\mathcal{L}_K$ denotes the Lie derivative of tensors. Those include the so-called generalized Robertson–Walker spacetimes, i.e., warped products

$$\bar{M}^{n+1} = I \times_f F^n$$

with warping function $f : I \to \mathbb{R}$, basis $I \subset \mathbb{R}$ an open interval with metric $-\mathrm{d}t^2$, and Riemannian fiber F^n. In this case, the conformal vector field $K = f\frac{\partial}{\partial t}$ is closed, in the sense that its dual 1-form ω^K is closed.

Generalized Robertson–Walker spacetimes include the usual models of simply connected spacetimes of sectional curvatures respectively equal to -1, 0 and 1, namely, the anti-De Sitter space $\mathbb{L}_{-1}^{n+1}$, the Lorentz–Minkwoski space $\mathbb{L}^{n+1}$ and the De Sitter space $\mathbb{S}_1^{n+1}$. A detailed account of the structure of conformally stationary spacetimes, as well as generalized Robertson–Walker spacetimes, can be found in [3,17].

Let $x : M^n \to \overline{M}^{n+1}$ denote a spacelike hypersurface of the $(n+1)$-dimensional, time-oriented Lorentz manifold $\overline{M}^{n+1}$. It is a standard fact that in this case M is orientable (see [18]), and if $K \in \mathcal{X}(\overline{M})$ time-orients $\overline{M}$, an orientation for M is given by a timelike unit normal vetor field N, globally defined on it, whose time orientation agrees with that of K. If A denotes the second fundamental form of x with respect to such a field N, and $\overline{M}^{n+1}$ has constant sectional curvature c, we recall Gauss' and Codazzi's equations: for $W, X, Y, Z \in \mathcal{X}(M)$, one has

$$\langle R(W,X)Y,Z\rangle = c[\langle W,Y\rangle\langle X,Z\rangle - \langle W,Z\rangle\langle X,Y\rangle] - \langle AW,Y\rangle\langle AX,Z\rangle + \langle AW,Z\rangle\langle AX,Y\rangle \tag{5}$$

and

$$(\nabla_X A)Y = (\nabla_Y A)X. \tag{6}$$

Note that, in this case, Codazzi's Eq. (6) is exactly what it means for the second fundamental form A to be a Codazzi tensor.

2.3. Higher order mean curvatures

From now on, $x : M^n \to \overline{M}^{n+1}$ will always denote a spacelike hypersurface M of the time-oriented, $(n+1)$-dimensional Lorentz manifold $\overline{M}$. Associated to the second fundamental form A of x one has n invariants S_r, $1 \le r \le n$, given by the equality

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in M$ and $\{e_k\}$ is a basis of T_pM formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$. In particular

$$|A|^2 + 2S_2 = S_1^2.$$

The following lemma appears, in a slightly different form, in [2].

Lemma 2. *Let $x : M^n \to \overline{M}^{n+1}$ denote an isometric immersion. If S_2 is constant on M, then*

$$S_1^2(|\nabla A|^2 - |\nabla S_1|^2) \ge 2S_2|\nabla A|^2. \tag{7}$$

In particular, if $S_2 \ge 0$ then $|\nabla A|^2 - |\nabla S_1|^2 \ge 0$.

Proof. Let $p \in M$ and $\{e_k\}$ be a moving frame on a neighborhood $U \subset M$ of p, geodesic at p. Letting (h_{ij}) denote the matrix of A with respect to $\{e_k\}$, it follows from $S_1^2 = |A|^2 + 2S_2$ that $S_1^2 = \sum_{k,l} h_{kl}^2 + 2S_2$. Therefore, one has at p

$$S_1 e_i(S_1) = \sum_{k,l} h_{kl} h_{kli}.$$

One now uses Cauchy–Schwarz inequality to get

$$S_1^2 (e_i(S_1))^2 = \left(\sum_{k,l} h_{kl} h_{kli} \right)^2 \leq \left(\sum_{k,l} h_{kl}^2 \right) \left(\sum_{k,l} h_{kli}^2 \right) = |A|^2 \left(\sum_{k,l} h_{kli}^2 \right).$$

Adding the above inequalities for $1 \leq i \leq n$, one finally gets

$$S_1^2 |\nabla S_1|^2 \leq |A|^2 |\nabla A|^2 = (S_1^2 - 2S_2)|\nabla A|^2,$$

which is the desired inequality. If $S_2 \geq 0$, it follows that $S_1^2(|\nabla A|^2 - |\nabla S_1|^2) \geq 0$ on M. Defining $U = \{p \in M; S_1(p) \neq 0\}$, one gets $|\nabla A|^2 - |\nabla S_1|^2 \geq 0$ on U, and hence on $\overline{U}$. In $\overline{U}^c$, which is open, it follows from $2S_2 + |A|^2 = 0$ and $S_2 \geq 0$ that $A = 0$. Therefore $\nabla A = 0$, and thus $\nabla S_1 = 0$, on $\overline{U}^c$, so that we also have $|\nabla A|^2 - |\nabla S_1|^2 \geq 0$ there. □

If R denotes the scalar curvature of M, and $\overline{M}$ has constant sectional curvature c, it follows from Gauss' equation that

$$2S_2 = n(n-1)(c - R), \tag{8}$$

so that S_2 is constant on M if and only if R is constant on M. In fact, if $p \in M$ and $\{e_k\}$ be a basis of T_pM with $Ae_k = \lambda_k e_k$ for $1 \leq k \leq n$, then

$$\begin{aligned} R(p) &= \frac{2}{n(n-1)} \sum_{i<j} \langle R(e_i, e_j)e_i, e_j \rangle \\ &= \frac{2}{n(n-1)} \sum_{i<j} [c - \langle Ae_i, e_i \rangle \langle Ae_j, e_j \rangle + \langle Ae_i, e_j \rangle^2] \\ &= \frac{2}{n(n-1)} \left[\binom{n}{2} c - \sum_{i<j} \lambda_i \lambda_j \right] = c - \frac{2S_2(p)}{n(n-1)}. \end{aligned}$$

It is sometimes more convenient to work with the higher order mean curvatures H_r of the immersion x, defined for $0 \leq r \leq n$ by

$$H_r = (-1)^r \frac{S_r}{\binom{n}{r}} = \frac{\sigma_r(-\lambda_1, \ldots, -\lambda_n)}{\binom{n}{r}}. \tag{9}$$

Such functions satisfy a very useful set of algebraic inequalities, usually reffered to as Newton's inequalities. A proof of them for positive real numbers can be found in [13]. Here, we present a more general version of them, together with a sharp condition for equality. For the proof, recall that if a polynomial $f \in \mathbb{R}[X]$ has $k \geq 1$ real roots, then its derivative f' has at least $k-1$ real roots. In particular, if all roots of f are real, then the same is true of all roots of f'.

Proposition 1. *Let $n > 1$ be an integer, and $\lambda_1, \ldots, \lambda_n$ be real numbers. Define, for $0 \leq r \leq n$, $S_r = S_r(\lambda_i)$ as above, and $H_r = H_r(\lambda_i) = \binom{n}{r}^{-1} S_r(\lambda_i)$.*

(a) *For $1 \leq r < n$, one has $H_r^2 \geq H_{r-1}H_{r+1}$. Moreover, if equality happens for $r = 1$ or for some $1 < r < n$, with $H_{r+1} \neq 0$ in this case, then $\lambda_1 = \cdots = \lambda_n$.*
(b) *If $H_1, H_2, \ldots, H_r > 0$ for some $1 < r \leq n$, then $H_1 \geq \sqrt{H_2} \geq \sqrt[3]{H_3} \geq \cdots \geq \sqrt[r]{H_r}$. Moreover, if equality happens for some $1 \leq j < r$, then $\lambda_1 = \cdots = \lambda_n$.*
(c) *If, for some $1 \leq r < n$, one has $H_r = H_{r+1} = 0$, then $H_j = 0$ for all $r \leq j \leq n$. In particular, at most $r-1$ of the λ_i are different from zero.*

Proof. In order to prove (a) we use induction on the number $n > 1$ of real numbers. For $n = 2$, $H_1^2 \geq H_0 H_2$ is equivalent to $(\lambda_1 - \lambda_2)^2 \geq 0$, with equality if and only if $\lambda_1 = \lambda_2$. Suppose the inequalities true for $n-1$ real numbers, with equality when $H_{r+1} \neq 0$ if and only if all of them are equal. Given $n \geq 3$ real numbers $\lambda_1, \ldots, \lambda_n$, let

$$f(x) = (x + \lambda_1) \ldots (x + \lambda_n) = \sum_{r=0}^{n} \binom{n}{r} H_r(\lambda_i) x^{n-r}.$$

Then

$$f'(x) = \sum_{r=0}^{n-1} (n-r) \binom{n}{r} H_r(\lambda_i) x^{n-r-1}.$$

On the other hand, there exist real numbers $\gamma_1, \ldots, \gamma_{n-1}$ such that

$$\begin{aligned} f'(x) &= n(x + \gamma_1) \cdots (x + \gamma_{n-1}) = n \sum_{r=0}^{n-1} S_r(\gamma_i) x^{n-1-r} \\ &= \sum_{r=0}^{n-1} n \binom{n-1}{r} H_r(\gamma_i) x^{n-1-r}. \end{aligned}$$

Since $(n-r)\begin{pmatrix} n \\ r \end{pmatrix} = n \begin{pmatrix} n-1 \\ r \end{pmatrix}$, comparing coefficients gives us $H_r(\lambda_i) = H_r(\gamma_i)$ for $0 \le r \le n-1$. Hence, it follows from the induction hypothesis that, for $1 \le r \le n-2$,

$$H_r^2(\lambda_i) = H_r^2(\gamma_i) \ge H_{r-1}(\gamma_i)H_{r+1}(\gamma_i) = H_{r-1}(\lambda_i)H_{r+1}(\lambda_i).$$

Moreover, if equality happens for the λ_i, with $H_{r+1}(\lambda_i) \ne 0$, then it will also happen for the γ_i, with $H_{r+1}(\gamma_i) \ne 0$. Again from the induction hypothesis, it follows that $\gamma_1 = \cdots = \gamma_{n-1}$, and thus $\lambda_1 = \cdots = \lambda_n$.

To finish, it suffices to prove that $H_{n-1}^2(\lambda_i) \ge H_{n-2}(\lambda_i)H_n(\lambda_i)$, with equality for $H_n \ne 0$ if and only if all of the λ_i are equal. If $\lambda_i = 0$ for some $1 \le i \le n$, equality is obvious. If not, $H_n \ne 0$ and

$$\begin{aligned} H_{n-1}^2 \ge H_{n-2}H_n \Leftrightarrow & \left[\begin{pmatrix} n \\ n-1 \end{pmatrix}^{-1} \sum_i \frac{H_n}{\lambda_i} \right]^2 \\ & \ge \left[\begin{pmatrix} n \\ n-2 \end{pmatrix}^{-1} \sum_{i<j} \frac{H_n}{\lambda_i \lambda_j} \right] H_n \Leftrightarrow (n-1)\left(\sum_i \frac{1}{\lambda_i} \right)^2 \ge 2n \sum_{i<j} \frac{1}{\lambda_i \lambda_j}. \end{aligned}$$

Denoting $\alpha_i = 1/\lambda_i$, the last inequality above is equivalent to

$$(n-1)\left(\sum_{i=1}^n \alpha_i \right)^2 \ge 2n \sum_{i<j} \alpha_i \alpha_j.$$

Letting $T(\alpha_i) = (n-1)\left(\sum_{i=1}^n \alpha_i\right)^2 - 2n\sum_{i<j}\alpha_i\alpha_j$, we get

$$\begin{aligned} T(\alpha_i) &= n\left(\sum_{i=1}^n \alpha_i \right)^2 - \left(\sum_{i=1}^n \alpha_i \right)^2 - 2n \sum_{i<j} \alpha_i \alpha_j \\ &= n\left[\left(\sum_{i=1}^n \alpha_i \right)^2 - 2\sum_{i<j} \alpha_i \alpha_j \right] - \left(\sum_{i=1}^n \alpha_i \right)^2 = n\sum_{i=1}^n \alpha_i^2 - \left(\sum_{i=1}^n \alpha_i \right)^2 \ge 0, \end{aligned}$$

by Cauchy–Schwarz inequality. Also, in this case equality happens if and only if all of the α_i (and then all of the λ_i) are equal. Note that the above reasoning also proves that $H_1^2 = H_2$ if and only if all of the λ_i are equal, for $T(\lambda_i) = n^2(n-1)[H_1^2(\lambda_i) - H_2(\lambda_i)]$.

Regarding (b), observe that $H_1 \ge H_2^{1/2}$ follows from (a). On the other hand, if $H_1 \ge H_2^{1/2} \ge \cdots \ge H_k^{1/k}$ for some $2 \le k < r$, then

$$H_k^2 \ge H_{k-1}H_{k+1} \ge H_k^{\frac{k-1}{k}} H_{k+1},$$

or still $H_k^{1/k} \geq H_{k+1}^{1/(k+1)}$. It now follows immediately from the above inequalities that, if $H_k^{1/k} = H_{k+1}^{1/(k+1)}$ for some $1 \leq k < r$, then $H_k^2 = H_{k-1}H_{k+1}$. Therefore, item (a) gives $\lambda_1 = \cdots = \lambda_n$.

To prove (c) suppose, without loss of generality, $r < n-1$. Since $H_r = H_{r+1} = 0$, one has equality in Newton's inequality

$$H_{r+1}^2 \geq H_r H_{r+2}.$$

If $H_{r+2} \neq 0$, it follows from (a) that $\lambda_1 = \cdots = \lambda_n = \lambda$. Hence, $H_r = 0 \Rightarrow \lambda = 0$, from where $H_{r+2} = 0$, a contradiction. Therefore $H_{r+2} = 0$, and analogously $H_j = 0$ for all $r \leq j \leq n$. To finish, it suffices to note that the polynomial $f(x)$ of item (a) is, in this case, just

$$f(x) = \sum_{j=0}^{n} S_j x^{n-j} = \sum_{j=0}^{r-1} S_j x^{n-j}. \quad \square$$

2.4. Newton transformations

Back to spacelike hypersurfaces $x : M^n \to \overline{M}^{n+1}$, for $0 \leq r \leq n$ one defines the r-th Newton transformation P_r on M by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = (-1)^r S_r I + A P_{r-1}.$$

A trivial induction shows that

$$P_r = (-1)^r (S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r),$$

so that Cayley–Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM, diagonalizing A at $p \in M$, also diagonalize all of the P_r at p. Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^{\perp} \subset T_pM$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq n \\ j_1, \ldots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

With the above notations, it is also immediate to check that $P_r e_i = (-1)^r S_r(A_i) e_i$, so that, according to [5],

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$.
(b) $\mathrm{tr}(P_r) = (-1)^r \sum_{i=1}^n S_r(A_i) = (-1)^r (n-r) S_r$.
(c) $\mathrm{tr}(AP_r) = (-1)^r \sum_{i=1}^n \lambda_i S_r(A_i) = (-1)^r (r+1) S_{r+1}$.
(d) $\mathrm{tr}(A^2 P_r) = (-1)^r \sum_{i=1}^n \lambda_i^2 S_r(A_i) = (-1)^r (S_1 S_{r+1} - (r+2) S_{r+2})$.

The following proposition, due to J. Hounie and M.L. Leite (Lemma 1.1 and Eq. (1.3) in [14], as well as proposition 1.5 of [15]), will be quite useful in the next section.

Proposition 2. *Let M be a Riemannian manifold, $x : M^n \to \bar{M}^{n+1}$ an isometric immersion and $p \in M$. If $S_r(p) = 0$, then*:

(a) *P_{r-1} is semi-definite at p.*
(b) *If $S_{r+1}(p) \neq 0$, then P_{r-1} is definite at p.*

Associated to each Newton transformation P_r one has the second order differential operator $L_r : \mathcal{D}(M) \to \mathcal{D}(M)$, given by

$$L_r(f) = \mathrm{tr}(P_r \operatorname{Hess} f).$$

When $\bar{M}^{n+1}$ is a constant sectional curvature Riemannian space, it was proved by H. Rosenberg in [19] that

$$L_r(f) = \mathrm{div}(P_r \nabla f),$$

where div stands for the divergence of a vector field on M. His proof also works for Lorentz ambient spaces $\bar{M}^{n+1}$; it suffices to use Lemma 5 below, instead of its Riemannian couterpart. Therefore, for $f, g \in \mathcal{D}(M)$, it follows from the properties of the divergence of vector fields that

$$L_r(fg) = f L_r(g) + g L_r(f) + 2\langle P_r \nabla f, \nabla g\rangle. \tag{10}$$

The following lemma is due to R. Reilly (see [20]). For the sake of completeness, as well as to set some useful notation, we include a short proof of it.

Lemma 3. *If (h_{ij}) denote the matrix of A with respect to a certain basis $\beta = \{e_k\}$ of $T_p M$ (not necessarily orthogonal), then the matrix (h_{ij}^r) of P_r with respect to the same basis is given by*

$$h_{ij}^r = \frac{(-1)^r}{r!} \sum_{i_k, j_k = 1}^{n} \epsilon_{i_1 \ldots i_r i}^{j_1 \ldots j_r j} h_{j_1 i_1} \ldots h_{j_r i_r}, \tag{11}$$

where

$$\epsilon_{i_1\ldots i_r}^{j_1\ldots j_r} = \begin{cases} \mathrm{sgn}(\sigma) & \text{, if the } i_k \text{ are pairwise distinct and} \\ & \sigma = (j_k) \text{ form a permutation of them;} \\ 0 & \text{, else.} \end{cases}$$

Proof. Recall that $P_r = (-1)^r \sum_{j=0}^{n} (-1)^j S_{r-j} A^j$, with the coefficients S_{r-j} not depending of the chosen basis of T_pM. Thus, it suffices to verify the above formula for a basis $\{e_k\}$ of T_pM, diagonalizing A at p, with $Ae_k = \lambda_k e_k$ for $1 \le k \le n$. In this case, the right hand side of (11) successively equals

$$\begin{aligned} &\frac{(-1)^r}{r!} \sum_{i_k, j_k = 1}^{n} \epsilon_{i_1\ldots i_r i}^{j_1\ldots j_r j} \delta_{j_1 i_1} \ldots \delta_{j_r i_r} \lambda_{j_1} \ldots \lambda_{j_r} \\ &= \frac{(-1)^r}{r!} \sum_{i_k \neq i} \epsilon_{i_1\ldots i_r i}^{i_1\ldots i_r j} \lambda_{i_1} \ldots \lambda_{i_r} = (-1)^r \delta_{ij} \sum_{\substack{i_1 < \cdots < i_r \\ i_k \neq i}} \lambda_{i_1} \ldots \lambda_{i_r} \\ &= \delta_{ij} (-1)^r S_r(A_i) = \langle P_r e_i, e_j \rangle = h_{ij}^r. \quad \square \end{aligned}$$

We use the above lemma to compute first derivatives of h_{ij}^r:

Lemma 4. *Let $\{e_k\}$ be a moving frame on a neighborhood of $p \in M$, diagonalizing the second fundamental form A at p, with $Ae_k = \lambda_k e_k$ for $1 \le k \le n$. Then, for $1 \le i, j \le n$, $i \neq j$, one has at p*

$$e_k(h_{ii}^r) = (-1)^r \sum_{l \neq i} S_{r-1}(A_{il}) h_{ll;k} \tag{12}$$

and

$$e_k(h_{ij}^r) = (-1)^{r+1} S_{r-1}(A_{ij}) h_{ij;k}, \tag{13}$$

where A_{ij} denotes the restriction of A to $\{e_i, e_j\}^{\perp} \subset T_pM$.

Proof. Forgetting for the moment the restriction of being $i \neq j$, it follows from (11) that

$$\begin{aligned} e_k(h_{ij}^r) = \frac{(-1)^r}{r!} \sum_{i_k, j_k = 1}^{n} \epsilon_{i_1\ldots i_r i}^{j_1\ldots j_r j} h_{j_1 i_1;k} h_{j_2 i_2} \ldots h_{j_r i_r} + \cdots + \frac{(-1)^r}{r!} \\ \times \sum_{i_k, j_k = 1}^{n} \epsilon_{i_1\ldots i_r i}^{j_1\ldots j_r j} h_{j_1 i_1} \ldots h_{j_{r-1} i_{r-1}} h_{j_r i_r;k}. \end{aligned} \tag{14}$$

At p, the first summand at the right hand side of (14) equals

$$\frac{(-1)^r}{r!}\sum_{i_k,j_k=1}^{n}\epsilon_{i_1\dots i_r i}^{j_1\dots j_r j}\delta_{i_2 j_2}\dots\delta_{i_r j_r}h_{j_1 i_1;k}\lambda_{i_2}\dots\lambda_{i_r}$$
$$=\frac{(-1)^r}{r!}\sum_{i_k,j_1=1}^{n}\epsilon_{i_1 i_2\dots i_r i}^{j_1 i_2\dots i_r j}h_{j_1 i_1;k}\lambda_{i_2}\dots\lambda_{i_r}. \tag{15}$$

Now, consider two cases separately: for $i=j$, (15)

$$=\frac{(-1)^r}{r!}\sum_{i_k,j_1=1}^{n}\epsilon_{i_1 i_2\dots i_r i}^{j_1 i_2\dots i_r i}h_{j_1 i_1;k}\lambda_{i_2}\dots\lambda_{i_r}=\frac{(-1)^r}{r!}\sum_{1\le i_k\le n}\epsilon_{i_1 i_2\dots i_r i}^{i_1 i_2\dots i_r i}h_{i_1 i_1;k}\lambda_{i_2}\dots\lambda_{i_r}$$
$$=\frac{(-1)^r}{r}\sum_{l\neq i}\sum_{\substack{i_2<\dots<i_r\\ i_k\neq i,l}}h_{ll;k}\lambda_{i_2}\dots\lambda_{i_r}=\frac{(-1)^r}{r}\sum_{l\neq i}S_{r-1}(A_{il})h_{ll;k}.$$

Since the same is true for all of the other summands, one gets

$$e_k(h_{ii}^r)=(-1)^r\sum_{l\neq i}S_{r-1}(A_{il})h_{ll;k}.$$

For $i\neq j$, it follows from the very definition of $\epsilon_{i_1 i_2\dots i_r i}^{j_1 i_2\dots i_r j}$ that (15)

$$=\frac{(-1)^r}{r!}\sum_{i_k\neq i,j}\epsilon_{j i_2\dots i_r i}^{i i_2\dots i_r j}h_{ij;k}\lambda_{i_2}\dots\lambda_{i_r}=-\frac{(-1)^r}{r}\sum_{\substack{i_k\neq i,j\\ i_2<\dots<i_r}}h_{ij;k}\lambda_{i_2}\dots\lambda_{i_r}$$
$$=\frac{(-1)^{r+1}}{r}S_{r-1}(A_{ij})h_{ij;k},$$

and (14) gives

$$e_k(h_{ij}^r)=(-1)^{r+1}S_{r-1}(A_{ij})h_{ij;k}.\quad\square$$

3. A formula for $L_r(S_r)$

From now on, $x : M^n \to \bar{M}_c^{n+1}$ denotes a spacelike hypersurface of the time-oriented Lorentz manifold $\bar{M}$, of constant sectional curvature c. We assume M oriented by the choice of a unit normal vector field N, globally defined on it, and let A denote the corresponding second fundamental form.

Proposition 3. *Let $x : M^n \to \bar{M}_c^{n+1}$ be as above, and $0 \le q < n$, $0 < r < n$. If $\{e_k\}$ is any orthonormal frame on M, then*

$$L_q(S_r) = (-1)^{r+q-1}L_{r-1}(S_{q+1}) + (-1)^{r-1}\sum_k \mathrm{tr}\{[P_q(\nabla_{e_k}P_{r-1}) - P_{r-1}(\nabla_{e_k}P_q)](\nabla_{e_k}A)\} + (-1)^{r-1}c[\mathrm{tr}(AP_{r-1})\mathrm{tr}(P_q) - \mathrm{tr}(P_{r-1})\mathrm{tr}(AP_q)] + (-1)^r\mathrm{tr}(A^2P_{r-1})\mathrm{tr}(AP_q) - (-1)^r\mathrm{tr}(AP_{r-1})\mathrm{tr}(A^2P_q). \quad (16)$$

Proof. Firstly, note that the validity of (16) does not depend on the particular chosen frame $\{e_k\}$. Let then $p \in M$ and $\{e_k\}$ be a moving frame on a neighborhood $U \subset M$ of p, diagonalizing A at p, with $Ae_k = \lambda_k e_k$ for $1 \le k \le n$. Denote by h_{ij} and h^r_{ij}, respectively, the components of A and P_r with respect to such a frame. It follows from Eq. (11) that

$$h^r_{ii} = \frac{(-1)^r}{r!}\sum_{i_k, j_k=1}^{n} \epsilon^{j_1\dots j_r i}_{i_1\dots i_r i} h_{j_1 i_1}\dots h_{j_r i_r} = \frac{(-1)^r}{r!}\sum_{i_k \ne i, \sigma=(j_k)} \mathrm{sgn}(\sigma) h_{j_1 i_1}\dots h_{j_r i_r}$$
$$= (-1)^r \sum_{\substack{i_1<\cdots<i_r\\ i_k\ne i}}\sum_{\sigma=(j_k)} \mathrm{sgn}(\sigma) h_{j_1 i_1}\dots h_{j_r i_r} = (-1)^r \sum_{\substack{i_1<\cdots<i_r\\ i_k\ne i}} A(c_{i_1},\dots,c_{i_r}), \quad (17)$$

where by $A(c_{i_1},\dots,c_{i_r})$ we mean the $r \times r$ determinant minor of A, obtained by choosing lines and columns of A with indices $i_1 < \cdots < i_r$. Hence,

$$S_r = \frac{(-1)^r}{n-r}\mathrm{tr}(P_r) = \frac{1}{n-r}\sum_i \sum_{\substack{i_1<\cdots<i_r\\ i_k\ne i}} A(c_{i_1},\dots,c_{i_r}) = \sum_{i_1<\cdots<i_r} A(c_{i_1},\dots,c_{i_r}),$$

for once one has chosen $1 \le i_1 < \cdots < i_r \le n$, there will be left $n - r$ possible choices for i in $\{1,\dots,n\}$. Since determinants are multilinear functions of their columns, one gets

$$e_k(S_r) = \sum_{i_1<\cdots<i_r} [A(c_{i_1;k}, c_{i_2},\dots,c_{i_r}) + \cdots + A(c_{i_1},\dots,c_{i_{r-1}}, c_{i_r;k})] \quad (18)$$

on U. At p, one has

$$A(c_{i_1;k}, c_{i_2},\dots,c_{i_r}) = \begin{vmatrix} h_{i_1 i_1;k} & 0 & \cdots & 0 \\ h_{i_2 i_1;k} & \lambda_{i_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_r i_1;k} & 0 & \cdots & \lambda_{i_r} \end{vmatrix} = h_{i_1 i_1;k}\lambda_{i_2}\dots\lambda_{i_r},$$

and analogously for the other summands, so that

$$e_k(S_r) = \sum_{i_1<\cdots<i_r} (h_{i_1 i_1;k}\lambda_{i_2}\dots\lambda_{i_r} + \cdots + \lambda_{i_1}\dots\lambda_{i_{r-1}} h_{i_r i_r;k}) = \sum_{i=1}^{n} h_{ii;k} S_{r-1}(A_i). \quad (19)$$

The last equality follows from the fact that, for fixed $1 \le i \le n$, $h_{ii;k}$ appears in the above sum together with all products $\lambda_{j_1}\cdots\lambda_{j_{r-1}}$, with $j_1,\dots,j_{r-1} \ne i$ (note that the above formula for $e_k(S_r)$ could have been obtained directly from (12). This alternative approach was chosen to ease, in what comes next, the computation of second derivatives).

To compute second derivatives, suppose further $\{e_k\}$ to be geodesic at p. It follows from (18) that

$$e_k(e_k(S_r)) = \sum_{i_1<\cdots<i_r} [A(c_{i_1;kk}, c_{i_2}, \ldots, c_{i_r}) + \cdots + A(c_{i_1}, \ldots, c_{i_{r-1}}, c_{i_r;kk})] + \sum_{s\neq t} \sum_{i_1<\cdots<i_r} A(c_{i_1}, \ldots, c_{i_s;k}, \ldots, c_{i_t;k}, \ldots, c_{i_r}),$$

and one gets at p

$$e_k(e_k(S_r)) = \sum_{i_1<\cdots<i_r} (h_{i_1i_1;kk}\lambda_{i_2}\ldots\lambda_{i_r} + \cdots + \lambda_{i_1}\ldots\lambda_{i_{r-1}}h_{i_ri_r;kk}) + \sum_{\substack{i_1<\cdots<i_r\\ s\neq t}} (h_{i_si_s;k}h_{i_ti_t;k} - h_{i_si_t;k}h_{i_ti_s;k})\lambda_{i_1}\ldots\hat{\lambda}_{i_s}\ldots\hat{\lambda}_{i_t}\ldots\lambda_{i_r}$$

Grouping equal occurrences of $(r-2)$-tuples $i_1 < \cdots < i_{r-2}$ in the last expression above, $e_k(e_k(S_r))$ equals

$$\sum_i \sum_{\substack{i_1<\cdots<i_{r-1}\\ i_k\neq i}} h_{ii;kk}\lambda_{i_1}\ldots\lambda_{i_{r-1}} + \sum_{i\neq j} \sum_{\substack{i_1<\cdots<i_{r-2}\\ i_k\neq i,j}} [h_{ii;k}h_{jj;k} - h^2_{ij;k}]\lambda_{i_1}\ldots\lambda_{i_{r-2}},$$

and finally

$$e_k(e_k(S_r)) = \sum_i S_{r-1}(A_i)h_{ii;kk} + \sum_{i\neq j} S_{r-2}(A_{ij})[h_{ii;k}h_{jj;k} - h^2_{ij;k}].$$

Therefore, we get at p

$$\begin{aligned}
L_q(S_r) = \operatorname{tr}(P_q \operatorname{Hess}(S_r)) &= \sum_{k=1}^{n}(-1)^q S_q(A_k)e_k(e_k(S_r)) \\
&= \sum_{i,k}(-1)^q S_q(A_k)S_{r-1}(A_i)h_{ii;kk} \\
&\quad + \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_q(A_k)S_{r-2}(A_{ij})[h_{ii;k}h_{jj;k} - h^2_{ij;k}] \\
&= \sum_i S_{r-1}(A_i)L_q(h_{ii}) + \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_q(A_k)S_{r-2}(A_{ij})h_{ii;k}h_{jj;k} \\
&\quad - \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_q(A_k)S_{r-2}(A_{ij})h^2_{ij;k}.
\end{aligned}$$

Lemma 1, as well as the remarks on commutation of indices in geodesic frames made right after it, allows one to conclude that, at p,

$$\begin{aligned}
&\sum_i S_{r-1}(A_i)L_q(h_{ii})\\
&\quad= \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)h_{iikk} = \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)h_{ikik}\\
&\quad= \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)(h_{ikik} - h_{ikki} + h_{ikki} - h_{kkii} + h_{kkii})\\
&\quad= \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)(h_{ikik} - h_{ikki}) + \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)h_{kkii}\\
&\quad= -\sum_{i,j,k}(-1)^q S_{r-1}(A_i)S_q(A_k)(h_{jk}R_{ijik} + h_{ij}R_{jkki})\\
&\qquad+ \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)h_{kkii}\\
&\quad= -\sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)\lambda_k R_{ikik} - \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)\lambda_i R_{ikki}\\
&\qquad+ \sum_k (-1)^{q+r-1} S_q(A_k)L_{r-1}(h_{kk}).
\end{aligned} \tag{20}$$

Now, write $r-1$ in place of q and $q+1$ in place of r in relation (20) to get

$$\begin{aligned}
L_{r-1}(S_{q+1}) &= \sum_i S_q(A_i)L_{r-1}(h_{ii}) + \sum_{\substack{i,j,k\\ i\neq j}}(-1)^{r-1}S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k}\\
&\quad- \sum_{\substack{i,j,k\\ i\neq j}}(-1)^{r-1}S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ij;k}^2.
\end{aligned} \tag{21}$$

Substituting the result of (20) into (21) we arrive at

$$\begin{aligned}
L_{r-1}(S_{q+1}) &= \sum_i (-1)^{r+q-1}S_{r-1}(A_i)L_q(h_{ii}) + \sum_{i,k}(-1)^{r-1}S_{r-1}(A_i)S_q(A_k)\lambda_k R_{ikik}\\
&\quad+ \sum_{i,k}(-1)^{r-1}S_{r-1}(A_i)S_q(A_k)\lambda_i R_{ikki}\\
&\quad+ \sum_{\substack{i,j,k\\ i\neq j}}(-1)^{r-1}S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k}\\
&\quad- \sum_{\substack{i,j,k\\ i\neq j}}(-1)^{r-1}S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ij;k}^2.
\end{aligned} \tag{22}$$

Finally, subtracting (22) from (20) gives

$$
\begin{aligned}
L_q(S_r) = (-1)^{r+q-1} L_{r-1}(S_{q+1}) &- \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)\lambda_k R_{ikik} \\
&- \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)\lambda_i R_{ikki} + \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_q(A_k)S_{r-2}(A_{ij})h_{ii;k}h_{jj;k} \\
&- \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_q(A_k)S_{r-2}(A_{ij})h_{ij;k}^2 - \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k} \\
&+ \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ij;k}^2. \qquad (23)
\end{aligned}
$$

In order to better examine the summands at the right hand side of (23), let

$$I = \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)\lambda_k R_{ikik}$$

and

$$II = \sum_{i,k}(-1)^q S_{r-1}(A_i)S_q(A_k)\lambda_i R_{ikki}.$$

Using Gauss' equation, one gets

$$
\begin{aligned}
I &= (-1)^{r-1}\sum_{i,k}\langle R(P_{r-1}e_i, P_q e_k)e_i, Ae_k\rangle \\
&= (-1)^{r-1}c\sum_{i,k}[\langle P_{r-1}e_i, e_i\rangle\langle P_q e_k, Ae_k\rangle - \langle P_{r-1}e_i, Ae_k\rangle\langle P_q e_k, e_i\rangle] \\
&\quad + (-1)^r\sum_{i,k}[\langle AP_{r-1}e_i, e_i\rangle\langle AP_q e_k, Ae_k\rangle - \langle AP_{r-1}e_i, Ae_k\rangle\langle AP_q e_k, e_i\rangle] \\
&= (-1)^{r-1}c\left[\mathrm{tr}(P_{r-1})\mathrm{tr}(AP_q) - \sum_k\langle AP_{r-1}e_k, P_q e_k\rangle\right] \\
&\quad + (-1)^r\mathrm{tr}(AP_{r-1})\mathrm{tr}(A^2P_q) - (-1)^r\sum_k\langle A^2P_{r-1}e_k, AP_q e_k\rangle \\
&= (-1)^{r-1}c[\mathrm{tr}(P_{r-1})\mathrm{tr}(AP_q) - \mathrm{tr}(AP_{r-1}P_q)] \\
&\quad + (-1)^r\mathrm{tr}(AP_{r-1})\mathrm{tr}(A^2P_q) - (-1)^r\mathrm{tr}(A^3P_{r-1}P_q)
\end{aligned}
$$

and

$$\begin{aligned}
II &= (-1)^{r-1}\sum_{i,k}\langle R(Ae_i, P_q e_k)e_k, P_{r-1}e_i\rangle \\
&= (-1)^{r-1}c\sum_{i,k}[\langle Ae_i, e_k\rangle\langle P_q e_k, P_{r-1}e_i\rangle - \langle Ae_i, P_{r-1}e_i\rangle\langle P_q e_k, e_k\rangle] \\
&\quad + (-1)^r\sum_{i,k}[\langle A^2 e_i, e_k\rangle\langle AP_q e_k, P_{r-1}e_i\rangle - \langle A^2 e_i, P_{r-1}e_i\rangle\langle AP_q e_k, e_k\rangle] \\
&= (-1)^{r-1}c\left[\sum_k\langle Ae_k, P_{r-1}P_q e_k\rangle - \mathrm{tr}(AP_{r-1})\mathrm{tr}(P_q)\right] \\
&\quad + (-1)^r\sum_k\langle A^2 e_k, AP_{r-1}P_q e_k\rangle - (-1)^r\mathrm{tr}(A^2 P_{r-1})\mathrm{tr}(AP_q) \\
&= (-1)^{r-1}c[\mathrm{tr}(AP_{r-1}P_q) - \mathrm{tr}(AP_{r-1})\mathrm{tr}(P_q)] \\
&\quad + (-1)^r\mathrm{tr}(A^3 P_{r-1}P_q) - (-1)^r\mathrm{tr}(A^2 P_{r-1})\mathrm{tr}(AP_q),
\end{aligned}$$

On the other hand, letting

$$III = \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_q(A_k)S_{r-2}(A_{ij})h_{ii;k}h_{jj;k} - \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_q(A_k)S_{r-2}(A_{ij})h_{ij;k}^2$$

and

$$IV = \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k} - \sum_{\substack{i,j,k\\ i\neq j}}(-1)^q S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ij;k}^2,$$

it follows from Lemma 4 that, at p,

$$\begin{aligned}
&\sum_{\substack{i,j,k\\ i\neq j}} S_q(A_k)S_{r-2}(A_{ij})h_{ii;k}h_{jj;k} \\
&\quad = \sum_{i,k} S_q(A_k)h_{ii;k}\sum_{j\neq i} S_{r-2}(A_{ij})h_{jj;k} = (-1)^{r-1}\sum_{i,k} S_q(A_k)h_{ii;k}e_k(h_{ii}^{r-1})
\end{aligned}$$

and

$$-\sum_{\substack{i,j,k\\ i\neq j}} S_q(A_k)S_{r-2}(A_{ij})h_{ij;k}^2 = (-1)^{r-1}\sum_{\substack{i,j,k\\ i\neq j}} S_q(A_k)h_{ij;k}e_k(h_{ij}^{r-1}).$$

Adding these two relations, one gets

$$III = (-1)^{r-1}\sum_{i,j,k} S_q(A_k)e_k(h_{ij}^{r-1})h_{ij;k} = (-1)^{r+q-1}\sum_k \mathrm{tr}[P_q(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)].$$

Again from Lemma 4, one has at p

$$\begin{aligned}
&-\sum_{\substack{i,j,k\\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k}\\
&\quad = -\sum_{i,k} S_{r-1}(A_k)h_{ii;k}\sum_{j\neq i} S_{q-1}(A_{ij})h_{jj;k} = -(-1)^q\sum_{i,k} S_{r-1}(A_k)h_{ii;k}e_k(h_{ii}^q)
\end{aligned}$$

and

$$\sum_{\substack{i,j,k\\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ij;k}^2 = -(-1)^q\sum_{\substack{i,j,k\\ i\neq j}} S_{r-1}(A_k)h_{ij;k}e_k(h_{ij}^q),$$

so that

$$IV = (-1)^q\sum_{i,j,k} S_{r-1}(A_k)e_k(h_{ij}^q)h_{ij;k} = -(-1)^{r+q-1}\sum_k \mathrm{tr}[P_{r-1}(\nabla_{e_k}P_q)(\nabla_{e_k}A)].$$

It now suffices to substitute the expressions for I, II, III and IV into (23). $\square$

As a byproduct of the computations in the above proof, we get the following

Lemma 5. *Let $x : M^n \to \bar{M}_c^{n+1}$ be an isometric immersion as described in the beginning of this section, and $V \in \mathcal{X}(M)$. Then*

$$\mathrm{tr}(P_{r-1}(\nabla_V A)) = (-1)^{r-1}V(S_r). \tag{24}$$

Proof. Let $p \in M$ and $\{e_k\}$ be a moving frame on a neighborhood of $p \in M$, geodesic at p and such that $Ae_k = \lambda_k e_k$ at p, for $1 \leq k \leq n$. Since both sides of (24) are linear in V, it suffices to prove that $\mathrm{tr}(P_{r-1}(\nabla_{e_k}A)) = (-1)^{r-1}e_k(S_r)$. But

$$\begin{aligned}
\mathrm{tr}(P_{r-1}(\nabla_{e_k}A)) &= \sum_{i=1}^n \langle P_{r-1}(\nabla_{e_k}A)e_i, e_i\rangle = \sum_{i=1}^n (-1)^{r-1}S_{r-1}(A_i)\langle(\nabla_{e_k}A)e_i, e_i\rangle\\
&= \sum_{i=1}^n (-1)^{r-1}S_{r-1}(A_i)h_{iik}.
\end{aligned}$$

Now, since the frame is geodesic at p, one gets $h_{iik} = h_{ii;k}$ at p, and (19) gives the desired result. $\square$

Corollary 1. *Let $x : M^n \to \bar{M}_c^{n+1}$ be an isometric immersion as set in the beginning of this section, and $0 < r \leq n$. Then*

$$\begin{aligned}L_r(S_r) = &-L_{r-1}(S_{r+1}) + S_r[(-1)^r \Delta S_r + L_{r-1}(S_1)]\\&+(-1)^r\left\{\sum_k |P_{r-1}\nabla_{e_k}A|^2 - |\nabla S_r|^2\right\} + \mathrm{tr}(AP_{r-1})\{S_r(|A|^2+cn)\\&-(-1)^r[\mathrm{tr}(A^2P_r)+c\,\mathrm{tr}(P_r)]\} + (-1)^r\mathrm{tr}(A^2P_{r-1})[\mathrm{tr}(A^2P_{r-1})+c\,\mathrm{tr}(P_{r-1})],\end{aligned} \tag{25}$$

where $\{e_k\}$ is any orthonormal frame on M, or still

$$\begin{aligned}L_r(S_r) = &-L_{r-1}(S_{r+1}) + S_r[(-1)^r \Delta S_r + L_{r-1}(S_1)]\\&+(-1)^r\left\{\sum_k |P_{r-1}\nabla_{e_k}A|^2 - |\nabla S_r|^2\right\}\\&+\frac{1}{2}\sum_{i,j}(-1)^r S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i-\lambda_j)^2 K_M(\sigma_{ij}),\end{aligned} \tag{26}$$

at $p \in M$, where $\{e_k\}$ is an orthonormal frame on M diagonalizing A at p, with $Ae_k = \lambda_k e_k$ at *p, and σ_{ij} denotes the 2-dimensional subspace of T_pM generated by e_i and e_j.*

Proof. It follows from Proposition 3 that

$$\begin{aligned}L_r(S_r) = &-L_{r-1}(S_{r+1}) + (-1)^{r-1}\sum_k \mathrm{tr}\left\{[P_r(\nabla_{e_k}P_{r-1}) - P_{r-1}(\nabla_{e_k}P_r)](\nabla_{e_k}A)\right\}\\&+(-1)^{r-1}c[\mathrm{tr}(AP_{r-1})\mathrm{tr}(P_r) - \mathrm{tr}(P_{r-1})\mathrm{tr}(AP_r)]\\&+(-1)^r\mathrm{tr}(A^2P_{r-1})\mathrm{tr}(AP_r) - (-1)^r\mathrm{tr}(AP_{r-1})\mathrm{tr}(A^2P_r),\end{aligned} \tag{27}$$

where $\{e_k\}$ is any orthonormal frame on M. Making

$$T_k = [P_r(\nabla_{e_k}P_{r-1}) - P_{r-1}(\nabla_{e_k}P_r)](\nabla_{e_k}A),$$

we get

$$\begin{aligned}T_k &= ((-1)^r S_r I + AP_{r-1})(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A) - P_{r-1}(\nabla_{e_k}(-1)^r S_r I + AP_{r-1})(\nabla_{e_k}A)\\&= (-1)^r S_r(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A) + AP_{r-1}(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A) - P_{r-1}[(-1)^r e_k(S_r)I\\&\quad+(\nabla_{e_k}A)P_{r-1} + A(\nabla_{e_k}P_{r-1})](\nabla_{e_k}A)\\&= (-1)^r S_r(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A) + (-1)^{r+1}e_k(S_r)P_{r-1}(\nabla_{e_k}A) - (P_{r-1}\nabla_{e_k}A)^2,\end{aligned}$$

so that

$$\begin{aligned}(-1)^{r-1}\sum_k \mathrm{tr}(T_k) = &-S_r\sum_k \mathrm{tr}[(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)] + \sum_k \mathrm{tr}[e_k(S_r)P_{r-1}(\nabla_{e_k}A)]\\&+(-1)^r\sum_k |P_{r-1}\nabla_{e_k}A|^2.\end{aligned} \tag{28}$$

Now, Lemma 5 gives

$$\sum_k \operatorname{tr}[e_k(S_r)P_{r-1}(\nabla_{e_k}A)] = \operatorname{tr}[P_{r-1}(\nabla_{\nabla S_r}A)] = (-1)^{r-1}|\nabla S_r|^2. \tag{29}$$

On the other hand, making $q = 0$ in Proposition 3 one gets

$$\begin{aligned}\Delta S_r &= (-1)^{r-1}L_{r-1}(S_1) + (-1)^{r-1}\sum_k \operatorname{tr}\left\{(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)\right\}\\ &= +(-1)^{r-1}c[\operatorname{tr}(AP_{r-1})n - \operatorname{tr}(P_{r-1})S_1]\\ &\quad + (-1)^r\operatorname{tr}(A^2P_{r-1})S_1 - (-1)^r\operatorname{tr}(AP_{r-1})|A|^2,\end{aligned}$$

so that

$$\begin{aligned}\sum_k \operatorname{tr}\left\{(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)\right\} &= (-1)^{r-1}\Delta S_r - L_{r-1}(S_1) - c[\operatorname{tr}(AP_{r-1})n - \operatorname{tr}(P_{r-1})S_1]\\ &\quad + \operatorname{tr}(A^2P_{r-1})S_1 - \operatorname{tr}(AP_{r-1})|A|^2.\end{aligned} \tag{30}$$

Substituting (29) and (30) into (28), and then into (27), we finally arrive at

$$\begin{aligned}L_r(S_r) = -L_{r-1}(S_{r+1}) &+ S_r[(-1)^r\Delta S_r + L_{r-1}(S_1)]\\ &+ (-1)^r\left\{\sum_k |P_{r-1}\nabla_{e_k}A|^2 - |\nabla S_r|^2\right\} + cS_r[\operatorname{tr}(AP_{r-1})n\\ &- \operatorname{tr}(P_{r-1})S_1] - S_r\operatorname{tr}(A^2P_{r-1})S_1 + S_r\operatorname{tr}(AP_{r-1})|A|^2\\ &+ (-1)^{r-1}c[\operatorname{tr}(AP_{r-1})\operatorname{tr}(P_r) - \operatorname{tr}(P_{r-1})\operatorname{tr}(AP_r)]\\ &+ (-1)^r\operatorname{tr}(A^2P_{r-1})\operatorname{tr}(AP_r) - (-1)^r\operatorname{tr}(AP_{r-1})\operatorname{tr}(A^2P_r),\end{aligned}$$

from where (25) easily follows. In order to get (26), let

$$\begin{aligned}T &= \operatorname{tr}(AP_{r-1})\{S_r(|A|^2 + cn) - (-1)^r[\operatorname{tr}(A^2P_r) + c\operatorname{tr}(P_r)]\}\\ &\quad - (-1)^{r-1}\operatorname{tr}(A^2P_{r-1})[\operatorname{tr}(A^2P_{r-1}) + c\operatorname{tr}(P_{r-1})]\end{aligned}$$

and take a basis $\{e_k\}$ of T_pM as in the statement of the corollary. Then

$$\begin{aligned}T &= \sum_i (-1)^{r-1}\lambda_i S_{r-1}(A_i)S_r(|A|^2 + cn) + \sum_{i,j}(-1)^r\lambda_i S_{r-1}(A_i)S_r(A_j)(c + \lambda_j^2)\\ &\quad + \sum_{i,j}(-1)^r\lambda_i^2 S_{r-1}(A_i)S_{r-1}(A_j)(c + \lambda_j^2)\\ &= \sum_i (-1)^{r-1}\lambda_i S_{r-1}(A_i)\cdot S_r(|A|^2 + cn)\\ &\quad + \sum_i (-1)^r\lambda_i S_{r-1}(A_i)\cdot\sum_j (c + \lambda_j^2)[S_r(A_j) + \lambda_i S_{r-1}(A_j)].\end{aligned}$$

Observing that

$$S_r(|A|^2+cn)-\sum_j(c+\lambda_j^2)[S_r(A_j)+\lambda_i S_{r-1}(A_j)]$$

$$=S_r(|A|^2+cn)-\sum_j(c+\lambda_j^2)[S_r+(\lambda_i-\lambda_j)S_{r-1}(A_j)]$$

$$=-\sum_j(c+\lambda_j^2)(\lambda_i-\lambda_j)S_{r-1}(A_j),$$

we get

$$T=\sum_{i,j}(-1)^r S_{r-1}(A_i)S_{r-1}(A_j)\lambda_i(\lambda_i-\lambda_j)(c+\lambda_j^2).$$

Doing the same computation as the one above, this time changing i per j from the very beginning, we arrive at

$$T=\sum_{i,j}(-1)^r S_{r-1}(A_j)S_{r-1}(A_i)\lambda_j(\lambda_j-\lambda_i)(c+\lambda_i^2),$$

so that

$$2T=\sum_{i,j}(-1)^r S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i-\lambda_j)[\lambda_i(c+\lambda_j^2)-\lambda_j(c+\lambda_i^2)]$$

$$=\sum_{i,j}(-1)^r S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i-\lambda_j)^2(c-\lambda_i\lambda_j),$$

$$=\sum_{i,j}(-1)^r S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i-\lambda_j)^2 K_M(\sigma_{ij})$$

where Gauss' equation was used in the last equality. □

Corollary 2. *Let $x: M^n \to \bar{M}_c^{n+1}$ be an isometric immersion as set in the beginning of this section. Then*

$$L_1(S_1)=-\Delta S_2-\left\{|\nabla A|^2-|\nabla S_1|^2\right\}$$
$$-\operatorname{tr}(AP_1)(|A|^2+cn)+S_1[\operatorname{tr}(A^2P_1)+c\operatorname{tr}(P_1)], \tag{31}$$

where $\{e_k\}$ is any orthonormal frame on M, or still

$$L_1(S_1)=-\Delta S_2-\left\{|\nabla A|^2-|\nabla S_1|^2\right\}-\frac{1}{2}\sum_{i,j}(\lambda_i-\lambda_j)^2K_M(\sigma_{ij}), \tag{32}$$

at p, where $\{e_k\}$ is an orthormal frame on M diagonalizing A at p, with $Ae_k = \lambda_k e_k$ at p, and σ_{ij} denotes the 2-dimensional subspace of T_pM generated by e_i and e_j;

$$L_2(S_2) = -L_1(S_3) - S_2\left\{|\nabla A|^2 - |\nabla S_1|^2\right\} + \sum_k |P_1\nabla_{e_k}A|^2 - |\nabla S_2|^2$$
$$+\operatorname{tr}(AP_2)[\operatorname{tr}(A^2P_1) + c\operatorname{tr}(P_1)] - \operatorname{tr}(AP_1)[\operatorname{tr}(A^2P_2) + c\operatorname{tr}(P_2)] \tag{33}$$

where $\{e_k\}$ is any orthonormal frame on M.

Proof. The first part of Corollary 2 is an immediate consequence of (25). For the second part, sustitute $r = 2$ in (25) to get

$$L_2(S_2) = -L_1(S_3) + S_2[\Delta S_2 + L_1(S_1)] + \sum_k |P_1\nabla_{e_k}A|^2 - |\nabla S_2|^2$$
$$+\operatorname{tr}(AP_1)\{S_2(|A|^2 + cn) - [\operatorname{tr}(A^2P_2) + c\operatorname{tr}(P_2)]\}$$
$$+\operatorname{tr}(A^2P_1)[\operatorname{tr}(A^2P_1) + c\operatorname{tr}(P_1)].$$

Now substitute, in the above formula, the expression for $\Delta S_2 + L_1(S_1)$, taken from the first part of the corollary. □

4. Applications

As in the previous section, by $x : M^n \to \overline{M}_c^{n+1}$ we mean a spacelike hypersurface of a time-oriented Lorentz manifold of constant sectional curvature c. Moreover, all spaces under consideration are supposed to be connected.

Theorem 1. *Let $x : M^n \to \overline{M}_c^{n+1}$ be a closed spacelike hypersurface of a time-oriented Lorentz manifold of constant sectional curvature $c \geq 0$. If M has constant scalar curvature R satisfying*

(a) $c\left(\frac{n-2}{n}\right) < R \leq c$, *then M is totally umbilical.*
(b) $c\left(\frac{n-2}{n}\right) \leq R \leq c$, *and* $S_3 \neq 0$, *then M is totally umbilical.*
(c) $c\left(\frac{n-2}{n}\right) \leq R < c$, *then* $\nabla A = 0$ *and M has constant mean curvature.*

Proof. It follows from Corollary 2 that

$$L_1(S_1) + |\nabla A|^2 - |\nabla S_1|^2 = 2S_2(|A|^2 + cn) - S_1[S_1S_2 - 3S_3 + (n-1)cS_1],$$

with

$$2S_2(|A|^2 + cn) - S_1[S_1S_2 - 3S_3 + (n-1)cS_1]$$
$$= 2S_2(S_1^2 - 2S_2 + cn) - S_1^2S_2 + 3S_1S_3 - c(n-1)S_1^2$$
$$= S_1^2S_2 - 4S_2^2 - c[(n-1)S_1^2 - 2nS_2] + 3S_1S_3. \tag{34}$$

Now, the first two of Newton's inequalities are respectively equivalent to

$$(n-1)S_1^2 \geq 2nS_2, \qquad 2(n-2)S_2^2 \geq 3(n-1)S_1S_3,$$

with equality happening at the first one, at a certain point of M, if and only if such a point is umbilical. Therefore, (34)

$$\begin{aligned} &\leq S_1^2S_2 - 4S_2^2 - c[(n-1)S_1^2 - 2nS_2] + \frac{2(n-2)S_2^2}{n-1} \\ &= S_1^2S_2 - \frac{2nS_2^2}{n-1} - c[(n-1)S_1^2 - 2nS_2] = [(n-1)S_1^2 - 2nS_2]\left(\frac{S_2}{n-1} - c\right). \end{aligned}$$

Taking (8) into account, condition $c\left(\frac{n-2}{n}\right) < R \leq c$ is equivalent to $0 \leq S_2 \leq (n-1)c$. Therefore,

$$L_1(S_1) + |\nabla A|^2 - |\nabla S_1|^2 \leq [(n-1)S_1^2 - 2nS_2]\left(\frac{S_2}{n-1} - c\right) \leq 0, \tag{35}$$

and integration over M gives

$$0 \leq \int_M \left\{|\nabla A|^2 - |\nabla S_1|^2\right\} \mathrm{d}M \leq \int_M [(n-1)S_1^2 - 2nS_2]\left(\frac{S_2}{n-1} - c\right) \mathrm{d}M \leq 0.$$

It follows that all of the above inequalities are in fact equalities, so that

$$2(n-2)S_2^2 = 3(n-1)S_1S_3, \qquad [(n-1)S_1^2 - 2nS_2]\left(\frac{S_2}{n-1} - c\right) = 0 \tag{36}$$

and, by Lemma 2,

$$|\nabla A|^2 - |\nabla S_1|^2 = 0.$$

Now, concerning (a), $\frac{S_2}{n-1} - c < 0$ gives $(n-1)S_1^2 = 2nS_2$, and M is totally umbilical. Also, if $S_3 \neq 0$ on M, the condition for equality in Proposition 1 assures, via (36), that M is totally umbilical. For (c), note that $S_2 \neq 0$. Then Lemma 2 gives $|\nabla A|^2 = 0$, and thus $\nabla S_1 = 0$. □

For the next result we need the following

Lemma 6. *Assume that the mean curvature H of M does not change sign, and choose the orientation of M in such a way that $H \geq 0$. If the scalar curvature R of M satisfies $R \leq c$, then $P_1 \geq 0$. If $R < c$ on M, then $P_1 > 0$ on M.*

Proof. It follows from (8) that $R \leq c$ if and only if $S_2 \geq 0$. Hence, letting $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the second fundamental form A of x, one has

$$S_1^2 = |A|^2 + 2S_2 \geq |A|^2 \geq \lambda_i^2. \tag{37}$$

Since $H \geq 0 \Leftrightarrow S_1 \leq 0$, one gets $S_1 \leq \lambda_i \leq -S_1$. Therefore, $S_1(A_i) = S_1 - \lambda_i \leq 0$, and $P_1 \geq 0$. If at some point $p \in M$ it happens that $S_1(A_i) = 0$, it follows from (37) that $S_2 = 0$ and $\lambda_j = 0$ for all $j \neq i$. Therefore, $S_1 \leq 0$ and $S_2 > 0$ give $P_1 > 0$. $\square$

Theorem 2. *Let $x : M^n \to \overline{M}_c^{n+1}$ be a complete spacelike hypersurface of a time-oriented Lorentz manifold of constant sectional curvature $c > 0$. Suppose that the mean curvature H of M does not change sign, and choose the orientation of M in such a way that $H \geq 0$. If H attains a global maximum on M, and M has constant scalar curvature R satisfying*

$$c\left(\frac{n-2}{n}\right) < R < c,$$

then M is totally umbilical.

Proof. Since $0 < \frac{S_2}{n-1} < c$, it follows from (35) and from Lemma 2 that

$$L_1(S_1) \leq [(n-1)S_1^2 - 2nS_2]\left(\frac{S_2}{n-1} - c\right) \leq 0.$$

By the preceding lemma L_1 is elliptic, and since S_1 attains a global minimum on M, Hopf's strong maximum principle assures that S_1 is constant on M. Thus,

$$[(n-1)S_1^2 - 2nS_2]\left(\frac{S_2}{n-1} - c\right) = 0$$

on M, from where it follows that $(n-1)S_1^2 - 2nS_2 = 0$ on M. The condition for equality in the first of Newton's inequalities now assures that M is totally umbilical. $\square$

For general r, Lemma 6 has the following substitute:

Lemma 7. *Let M be of Ricci curvature $\mathrm{Ric} \leq c$. Also, suppose that the mean curvature H of M does not change sign, and choose the orientation in such a way that $H \geq 0$. If $H_r(p) \neq 0$ for some $2 \leq r \leq n$, then L_{r-1} is elliptic at p.*

Proof. Fix $p \in M$ and choose a basis $\{e_k\}$ of T_pM, diagonalizing A at p, with $Ae_k = \lambda_k e_k$ for $1 \leq k \leq n$. Gauss' equation gives

$$\mathrm{Ric}_p(e_k) = \frac{1}{n-1}\sum_{i\neq k}(c - \lambda_k\lambda_i) = c - \frac{1}{n-1}\lambda_k(S_1 - \lambda_k).$$

Hence $\mathrm{Ric}_p(e_k) \leq c$ and $S_1(p) \leq 0$ give $-S_1(p) \leq \lambda_k \leq 0$ for $1 \leq k \leq n$. It follows that all of the summands in $H_r(p)$ are nonnegative, so that $H_r(p) \geq 0$. If $H_r(p) \neq 0$, then $H_r(p) > 0$ and at least r of the λ_k are negative, so that, at p, at least one of the summands of $(-1)^{r-1}S_{r-1}(A_i)$ is positive, for all $1 \leq i \leq n$. Therefore, P_{r-1} is positive definite at p. $\square$

Theorem 3. *Let $x : M^n \to \bar{M}_c^{n+1}$ be a closed spacelike hypersurface of a time-oriented Lorentz manifold of constant sectional curvature $c > 0$. If the sectional curvature K_M of M satisfies $0 \le K_M \le c$ and, for some $2 \le r < n$, $H_r \ne 0$ is constant on M, then M has second fundamental form parallel and definite. Moreover, if $0 < K_M \le c$ then M is totally umbilical.*

Proof. >From $K_M \le c$ it follows that M has Ricci curvature $\mathrm{Ric} \le c$. Moreover, letting $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the second fundamental form A of M, it also follows from $K_M \le c$ that, at each point of M, one has either $\lambda_1, \ldots, \lambda_n \ge 0$ or $\lambda_1, \ldots, \lambda_n \le 0$. Therefore, the mean curvature H of M does not change sign, for otherwise there would exist $p \in M$ for which $H(p) = 0$, so that $\lambda_1 = \cdots = \lambda_n = 0$ at p. This fact would contradict $H_r(p) \ne 0$. Therefore, orienting M in such a way that $H \ge 0$, Lemma 7 assures the ellipticity of L_{r-1}. Eq. (26) gives at $p \in M$

$$0 = (-1)^r L_{r-1}(S_1 S_r - S_{r+1}) + \sum_k |P_{r-1} \nabla_{e_k} A|^2 + \frac{1}{2} \sum_{i,j} (-1)^{r-1} S_{r-1}(A_i)(-1)^{r-1} S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}).$$

Since P_{r-1} is positive definite and $K_M \ge 0$, the last term at the right hand side of the above expression is nonnegative, so that

$$(-1)^r L_{r-1}(S_1 S_r - S_{r+1}) + \sum_k |P_{r-1} \nabla_{e_k} A|^2 \le 0.$$

Hence, $(-1)^r L_{r-1}(S_1 S_r - S_{r+1}) \le 0$ and, since M is closed and L_{r-1} is elliptic, Hopf's strong maximum principle guarantees that $S_1 S_r - S_{r+1}$ is constant on M. Therefore, $\sum_k |P_{r-1} \nabla_{e_k} A|^2 = 0$, and the definiteness of P_{r-1} gives $\nabla A = 0$.

Finally, it follows from

$$\sum_{i,j} (-1)^r S_{r-1}(A_i)(-1)^r S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}) = 0$$

that $(\lambda_i - \lambda_j)^2 (c - \lambda_i \lambda_j) = 0$ for all $1 \le i, j \le n$. This way, $\lambda_i(p) = 0$ for some $p \in M$ and some $1 \le i \le n$ gives $c\lambda_j^2 = 0$ for all $j \ne i$, so that $H_r(p) = 0$, a contradiction. This proves that the second fundamental form is definite. Moreover, $K_M > 0$ gives $(\lambda_i - \lambda_j)^2 = 0$ for all $1 \le i, j \le n$, and M is totally umbilical. □

Corollary 3. *Let $x : M^n \to \bar{M}_c^{n+1}$, $c > 0$, be a closed spacelike hypersurface of the time-oriented Lorentz manifold $\bar{M}$, of constant sectional curvature c. If the sectional curvature K_M of M satisfies $0 \le K_M \le c$ and, for some $2 \le r < n$, $H_r \ne 0$ is constant on M, then H_{r+1} is constant on M if and only if H (or the scalar curvature R) is constant on M.*

Proof. It follows from the previous result that $S_1S_r - S_{r+1}$ is constant on M. Therefore, S_{r+1} is constant on M if and only if S_1 is also constant on M. It now suffices to note that $2S_2 + |A|^2 = S_1^2$, and $\nabla A = 0 \Rightarrow |A|^2$ constant on M. $\square$

For generalized Robertson–Walker spacetimes, we get:

Corollary 4. *Let $\bar{M} = I \times_f F$ be a generalized Robertson-Walker spacetime of constant sectional curvature, and $x : M^n \to \bar{M}$ be a closed hypersurface of $\bar{M}$. If, for some $2 \le r < n$, one has $H_r \ne 0$ constant on M, and $0 < K_M \le c$, then $M = \{t\} \times F$, for some $t \in I$.*

Proof. Theorem 3 gives M totally umbilical. On the other hand, $H_r \ne 0$ on M gives $H \ne 0$ on M. Now applying a theorem of S. Montiel (theorem 6 of [17]), we get the desired result. $\square$

Corollary 5. *Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a closed spacelike hypersurface of the De Sitter space $\mathbb{S}_1^{n+1}$. If, for some $2 \le r < n$, $H_r \ne 0$ is constant on M, and $0 < K_M \le c$, then M is totally umbilical* (*and thus a round sphere*).

Remark 4. In [1] the authors got the above corollary assuming M entirely contained in the chronological future or past of an equator of the De Sitter space $\mathbb{S}_1^{n+1}$ (intead of being $0 < K_M \le c$). Afterwards, in [3], the authors generalized the above-mentioned result to generalized Robertson-Walker spacetimes of constant sectional curvature, obtaining Corollary 4 under the same change of hypotheses.

In what follows, we say that a spacelike hypersurface $x : M^n \to \bar{M}^{n+1}$, of a time-oriented Lorentz manifold $\bar{M}$, is *r-maximal* (maximal, if $r = 0$) when $H_{r+1} = 0$. The Calabi–Bernstein theorem (see [8]) assures that all maximal complete spacelike hypersurfaces of the Lorentz–Minkowsky space $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. The result is in fact more general, in the sense that the only maximal complete spacelike hypersurfaces of a time-oriented Lorentz manifold of constant sectional curvature $c \ge 0$ are the totally geodesic ones. In fact, making $r = 1$ in the first formula of Corollary 1, and using $|A|^2 + 2S_2 = S_1^2$ one gets

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + |A|^4 + nc|A|^2 \ge |A|^4,$$

and from this point on the proof is the same as that of the case $c = 0$.

In what follows, we present a weak extension of Calabi–Bernstein theorem for r-maximal spacelike hypersurfaces M^n of $\mathbb{L}^{n+1}$, which reduces to the above-mentioned theorem when $r = 0$. To this end, let $x : M^n \to \bar{M}^{n+1}$ be as before, with second fundamental form A. For $p \in M$, one defines the *space of relative nullity* $\Delta(p)$ of x at p by

$$\Delta(p) = \{v \in T_pM;\ v \in \mathrm{Ker}(A_p)\},$$

where Ker denotes the kernel of A_p. The *index of relative nullity* $\nu(p)$ of x at p is the dimension of $\Delta(p)$:

$\nu(p) = \dim(\Delta(p))$.

Theorem 4. *Let $x : M^n \to \overline{M}_c^{n+1}$ be a spacelike hypersurface of a time-oriented Lorentz manifold of constant sectional curvature $c \geq 0$. If $H_r = 0$ and H_{r+1} is constant on M, then $H_j = 0$ on M for all $r \leq j \leq n$, and*

(a) $\nu(p) \geq n - r + 1$ *for all $p \in M$.*
(b) *If $\overline{M}$ is the Lorentz–Minkowski space $\mathbb{L}^{n+1}$, and M is complete, then through every point of M there passes an $(n - r + 1)$-hyperplane of $\mathbb{L}^{n+1}$, totally contained in M.*

Proof. Let $\{e_k\}$ be any orthonormal moving frame on M. It follows from (25) that

$$\begin{aligned}
0 &= \sum_k |P_{r-1}\nabla_{e_k} A|^2 + \mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})] \\
&= \sum_k |P_{r-1}\nabla_{e_k} A|^2 + (r+1)S_{r+1}[(r+1)S_{r+1} - c(n-r+1)S_{r-1}] \\
&= \sum_k |P_{r-1}\nabla_{e_k} A|^2 + n^2 \binom{n-1}{r}^2 H_{r+1}^2 - n^2 \binom{n-1}{r}\binom{n-1}{r-1} cH_{r+1}H_{r-1}.
\end{aligned}$$

Now, Newton' inequalities give us $H_{r+1}H_{r-1} \leq H_r^2 = 0$, so that $-cH_{r+1}H_{r-1} \geq 0$. Therefore, all summands in the last line above are nonnegative, so that $H_{r+1} = 0$. By item (c) of Proposition 1, it follows that $H_j = 0$ on M for all $r \leq j \leq n$, so that the characteristic polynomial of A has, at each $p \in M$, at least $n - r + 1$ vanishing principal curvatures. Since the corresponding eigenvectors are linearly independent elements of $\Delta(p)$, (a) follows.

Letting ν_0 be the index of minimum relative nullity of M, we have $\nu_0 \geq n - r + 1$. Now, by theorem 5.3 of [10], the distribution $p \mapsto \Delta(p)$ of minimal relative nullity of M is smooth and integrable with complete leaves, totally geodesic in M and in $\overline{M}$. Therefore, item (b) follows from the characterization of complete totally geodesic submanifolds of the Lorentz–Minkowski space as spacelike hyperplanes of suitable dimension. □

Remark 5. The conclusion of item (b) above is, in a sense, the best one can get in the Lorentz–Minkowski space. In fact, let $M^n = M_1^{r-1} \times \mathbb{R}^{n-r+1}$, where M_1 is a complete $(r-1)$−dimensional Riemannian manifold. Then $\nu(p) \geq n - r + 1$ for all $p \in M$.

Corollary 6. *Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a complete spacelike hypersurface of the De Sitter space, with constant H_1, H_2 and H_3, $H_1 \neq 0$. If $H_2 = 0$ then M is a rotation hypersurface.*

Proof. By the previous result, it follows that $\nu(p) \geq n - 1$ for all $p \in M$. Therefore, proposition 1.2 of [7] guarantees that M is a rotation hypersurface. □

In the compact case we have a stronger result:

Theorem 5. *Let $x : M^n \to \overline{M}_c^{n+1}$ be a closed spacelike hypersurface of a time-oriented Lorentz manifold $\overline{M}$, of constant sectional curvature c. If $H_r = 0$ on M, then*

$$\int_M \mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})]\mathrm{d}M \leq 0 \tag{38}$$

and, moreover,

(a) $c \geq 0 \Rightarrow H_j = 0$ *on M, for all $r \leq j \leq n$.*
(b) *If $c \leq 0$ and $H_{r+1} \neq 0$, then*

$$\mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})] \geq 0$$

on M gives $\nabla A = 0$ and H_{r+1}, H_{r-1} constant on M.

Proof. Let $\{e_k\}$ be any orthonormal frame on M. It follows again from (25) that

$$L_{r-1}(S_{r+1}) = \sum_k |P_{r-1}\nabla_{e_k} A|^2 + \mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})].$$

Integrating over M, we get

$$\int_M \left\{ \sum_k |P_{r-1}\nabla_{e_k} A|^2 + \mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})] \right\} \mathrm{d}M = 0,$$

and so

$$\int_M \mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})]\mathrm{d}M \leq 0.$$

As in the proof of the previous result, one has for $c \geq 0$ that

$$\mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})] \geq 0$$

with equality if and only if $H_{r+1} = 0$. Therefore, it follows from (38) that $H_{r+1} = 0$, and item (c) of Proposition 1 gives $H_j = 0$ for $r \leq j \leq n$. This concludes the proof of item (a).

For $c \leq 0$ and $H_{r+1} \neq 0$, if $\mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})] \geq 0$ on M then (38) gives

$$\mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})] = 0 \tag{39}$$

on M. Hence,

$$\sum_k |P_{r-1}\nabla_{e_k} A|^2 = 0 \quad \text{and} \quad L_{r-1}(S_{r+1}) = 0$$

on M. Now, Proposition 2 assures that P_{r-1} is definite on M, so that $\nabla A = 0$. Moreover, integrating

$$\frac{1}{2}L_{r-1}(S_{r+1}^2) = \langle P_{r-1}\nabla S_{r+1}, \nabla S_{r+1}\rangle$$

over M gives H_{r+1} constant on M. Eq. (39) still gives, according to the proof of the preceding theorem, $H_{r+1} = \frac{nc}{n-r}H_{r-1}$, so that H_{r-1} is also constant on M. □

Remark 6. Specializing the previous result to conformally stationary Lorentz manifolds, observe that the statement of theorem 5 of [3] is incomplete. Asking H_{r-1} and H_r to be constant on M does not suffice to guarantee the umbilicity of M. According to Proposition 1, what is missing is the hypothesis $H_{r+1} \neq 0$.

Corollary 7. *Let $\overline{M}^{n+1}$ be a conformally stationary Lorentz manifold of constant sectional curvature $c \leq 0$, and $x : M^n \to \overline{M}_c^{n+1}$ a closed spacelike hypersurface with $H_r = 0$. If*

$$\mathrm{tr}(A^2 P_{r-1})[\mathrm{tr}(A^2 P_{r-1}) + c\,\mathrm{tr}(P_{r-1})] \geq 0,$$

on M, then there exists $p \in M$ such that $H_{r+1}(p) = 0$.

Proof. To the contrary, suppose that $H_{r+1} \neq 0$ on M for such an immersion. Then, according to the above theorem, H_{r-1} and H_{r+1} would be constant on M, with $H_{r+1} \neq 0$. Therefore, by theorem 7 of [3], M^n would be totally umbilical. Thus, letting λ be the umbilicity factor of M, it would follow from $H_r = 0$ that $\lambda = 0$, and from $H_{r+1} \neq 0$ that $\lambda \neq 0$, a contradiction. □

References

[1] J.A. Aledo, L.J. Alías, A. Romero, Integral formulas for compact spacelike hypersurfaces in the de sitter space: applications to the case of constant higher order mean curvature, J. Geom. Phys. 31 (1999) 195–208.

[2] H. Alencar, M. do Carmo, G. Colares, Stable hypersurfaces with constant scalar curvature, Math. Z. 213 (1993) 117–131.

[3] L.J. Alías, A. Brasil Jr., A.G. Colares, Integral formulae for spacelike hypersurfaces in conformally stationary spacetimes and applications, Proc. Edinburgh Math. Soc. 46 (2003) 465–488.

[4] L.J. Alías, A. Romero, M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized robertson-walker spacetimes, Gen. Relat. Gravit. 27 (1995) 71–84.

[5] J.L.M. Barbosa, A.G. Colares, Stability of hypersurfaces with constant r-mean curvature, Ann. Global Anal. Geom. 15 (1997) 277–297.

[6] J.L.M. Barbosa, V. Oliker, Spacelike hypersurfaces with constant mean curvature in lorentz spaces, Matem. Contemporânea 4 (1993) 27–44.

[7] A. Brasil Jr., A.G. Colares, O. Palmas, Complete spacelike hypersurfaces with constant mean curvature in the de sitter space: A gap theorem, Ill. J. Math. 47 (2003) 847–866.

[8] S.Y. Cheng, S.T. Yau, Maximal spacelike hypersurfaces in the lorentz–minkowski space, Ann. Math. 104 (1976) 407–419.

[9] S.Y. Cheng, S.T. Yau, Hypersurfaces with constant scalar curvature, Math. Ann. 225 (1977) 195–204.

[10] M. Dajczer, et al., Submanifolds and Isometric Immersions, Publish or Perish, Houston, 1990.

[11] A.J. Goddard, Some remarks on the existence of spacelike hypersurfaces of constant mean curvature, Math. Proc. Cambridge Phil. Soc. 82 (1977) 489–495.
[13] G. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge Mathematical Library, Cambridge, 1989.
[14] J. Hounie, M.L. Leite, The maximum principle for hypersurfaces with vanishing curvature functions, J. Diff. Geom. 41 (1995) 247–258.
[15] J. Hounie, M.L. Leite, Two-ended hypersurfaces with zero scalar curvature, Ind. Univ. Math. J. 48 (1999) 867–882.
[16] S. Montiel, An integral inequality for compact spacelike hypersurfaces in the de sitter space and applications to the case of constant mean curvature, Ind. Univ. Math. J. 37 (1988) 909–917.
[17] S. Montiel, Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes, Math. Ann. 314 (1999) 529–553.
[18] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.
[19] H. Rosenberg, Hypersurfaces of constant curvature in space forms, Bull. Sci. Math. 117 (1993) 217–239.
[20] R. Reilly, Variational properties of functions of the mean curvatures for hypersurfaces in space forms, J. Diff. Geom. 8 (1973) 465–477.
[21] J. Simons, Minimal varieties in riemannian manifolds, Ann. Math. 88 (1968) 62–105.